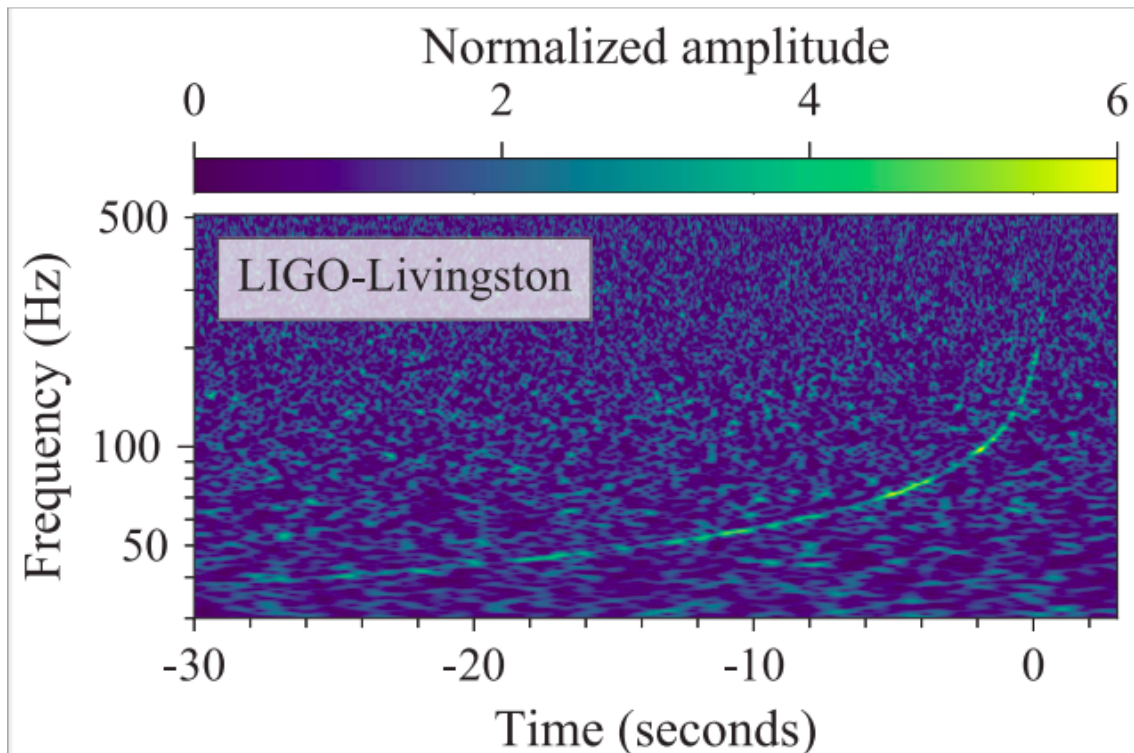


In this problem, we will analyze Time-frequency representations of data containing the gravitational-wave event GW170817, Which was the first observation by LIGO of a merger of a binary neutron star system.



1 Data from the graph

(1.1) You are given semi-logarithmic graph, from which you need to extract values of frequency and time. Find a linear expression to obtain actual time t from the measured horizontal axis coordinate x .

(2pt)

To make the scaling as accurate as possible, we take the time values far apart. When $x = 0.0 \text{ cm}$ $t = -30 \text{ s}$, using the ruler we can also measure that at $t = 0 \text{ s}$ we have $x = 11.1 \text{ cm}$. This leads to

1.0

$$\begin{aligned} 0.0 \text{ cm} &= (-30 \text{ s})A + B \\ 11.1 \text{ cm} &= (0 \text{ s})A + B = B \\ A &= \frac{11.1 \text{ cm}}{30 \text{ s}} = 0.37 \text{ cm/s} \\ \therefore t &= \frac{x - 11.1}{0.37} \text{ s} \end{aligned}$$

1.0

(1.2) Similarly, find an expression for the frequency f as a function of the measured coordinate y .

(3pt)

Analogues to the previous scaling,

$$0.0 \text{ cm} = C \log(30 \text{ Hz}) + D$$

$$5.3 \text{ cm} = C \log(500 \text{ Hz}) + D$$

$$5.3 \text{ cm} = C \log\left(\frac{500}{30}\right)$$

$$C = 4.34 \text{ cm}$$

$$D = -6.41 \text{ cm}$$

$$\therefore f = 10^{\left(\frac{y+6.41}{4.34}\right)} \text{ Hz}$$

1.0

2.0

(1.3) Using the relations that you obtained, extract at least 12 values of time and corresponding frequency from the given graph. At least one of the values should correspond to a frequency more than 100 Hz. (6pt)

It is not necessary to use X -scaling as students can choose convenient points from the graph.

| x(cm) | y(cm) | t(s) | f(Hz) | $f^{-8/3}(Hz^{-8/3})$ |
|-------|-------|-------|-------|-----------------------|
| - | 0.5 | -28.0 | 39.1 | 5.67E-05 |
| - | 0.5 | -26.0 | 39.1 | 5.67E-05 |
| - | 0.6 | -24.0 | 41.3 | 4.92E-05 |
| - | 0.6 | -22.0 | 41.3 | 4.92E-05 |
| - | 0.7 | -20.0 | 43.5 | 4.27E-05 |
| - | 0.8 | -18.0 | 45.9 | 3.71E-05 |
| - | 0.9 | -16.0 | 48.4 | 3.22E-05 |
| - | 1.0 | -14.0 | 51.0 | 2.79E-05 |
| - | 1.1 | -12.0 | 53.8 | 2.43E-05 |
| - | 1.2 | -10.0 | 56.7 | 2.11E-05 |
| - | 1.4 | -8.0 | 63.1 | 1.59E-05 |
| - | 1.6 | -6.0 | 70.1 | 1.20E-05 |
| - | 1.8 | -4.0 | 78.0 | 0.90E-05 |
| - | 2.3 | -2.0 | 101.7 | 0.43E-05 |
| - | 3.4 | 0.0 | 182.4 | 0.09E-05 |

0.5 points for each row (except

last column). Maximum 6 points.

2 Calculate system parameters

The most plausible explanation for this evolution of frequency is the in-spiralling of two orbiting masses, m_1 and m_2 , due to gravitational-wave emission. At the lower frequencies, such evolution is characterized by the chirp mass

$$M_{chirp} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} = \frac{c^3}{G} \left[\frac{5}{96} \pi^{-8/3} f^{-11/3} \dot{f} \right]^{3/5}$$

where f and \dot{f} are the observed frequency and its time derivative and G and c are the gravitational constant and speed of light.

(2.1) Linearize the equation given above and obtain the frequency dependence on time. (3pt)

Note: If $x^n \dot{x} = k$, then $\frac{x^{n+1}}{(n+1)} = kt + C$, where k , n and C are all some constants.

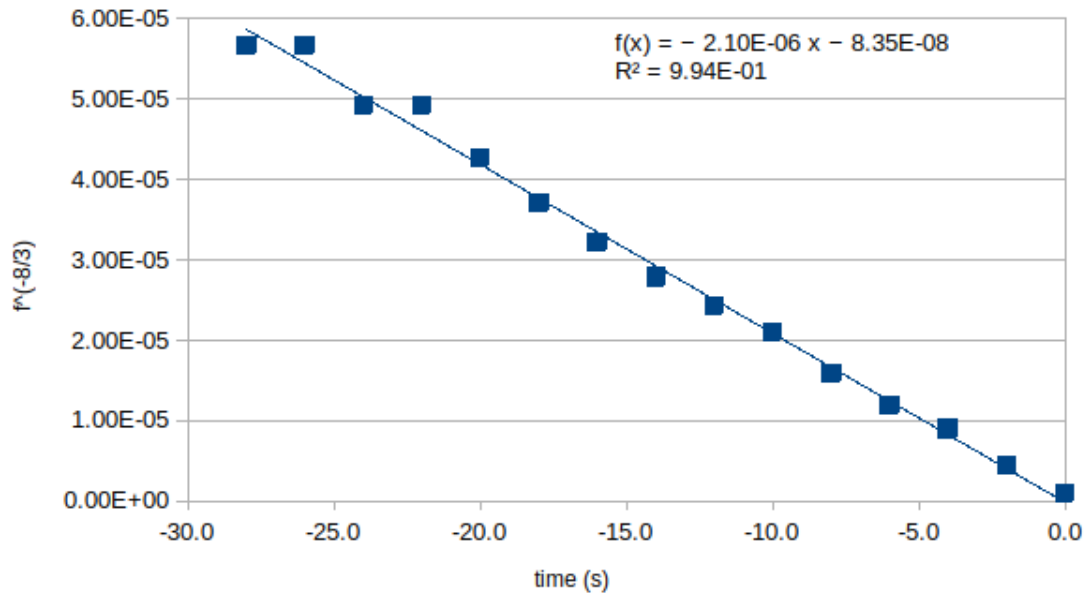
$$\frac{96}{5}\pi^{8/3}\left(\frac{GM_{chirp}}{c^3}\right)^{5/3} = f^{-11/3}\dot{f}$$

$$\therefore \left[\frac{96}{5}\pi^{8/3}\left(\frac{GM_{chirp}}{c^3}\right)^{5/3}\right]t + \left[\frac{3}{8}\right]f^{-8/3} + C = 0$$

3.0

(2.2) Find chirp mass in terms of solar masses and its uncertainty, by using a millimeter paper.

(15pt)



Calculation of $f^{-8/3}$ values
Properly drawn graph

3.0

6.0

$$\text{slope} = \frac{8}{3} \left[\frac{96}{5} \pi^{8/3} \left(\frac{GM_{chirp}}{c^3} \right)^{5/3} \right]$$

$$M_{chirp} = \left[\left(\frac{5}{256} \right)^{3/5} \frac{c^3}{G} \pi^{-8/5} \right] (\text{slope})^{3/5}$$

2.0

$$\text{slope} = 2.10 \times 10^{-6}$$

2.0

$$\therefore M_{chirp} = 2.39 \times 10^{30} \text{ kg} = 1.20 M_{\odot}$$

2.0

with 95% confidence interval

$$\Delta \text{slope} = 9 \times 10^{-8}$$

$$\frac{\Delta M_{chirp}}{M_{chirp}} = \frac{3}{5} \frac{\Delta \text{slope}}{\text{slope}} = 0.04$$

$$\Delta M_{chirp} = 0.03 M_{\odot}$$

We realise that what is measured by the ground-based GW detectors is actually the detector-frame masses, which are related to the source frame masses by

$$m_{\text{detector}} = (1 + z)m$$

where z is the redshift of the binary.

(2.3) It is known that host galaxy NGC 4993 has a red shift $z = 0.009783$, find the source frame chirp mass. (2pt)

$$M_{chirp} = \frac{M_{chirp_{\text{detector}}}}{1+z} = \frac{1.20M_{\odot}}{1.009783} = 1.19M_{\odot}$$

2.0

(2.4) Find distance to the NGC 4993. (3pt)

According to the Hubble law,

$$D = \frac{zc}{H} = \frac{0.009783 \times 3 \times 10^5}{70} = 41.9 \text{ Mpc}$$

2.0

(2.5) The mass ratio $q = m_1/m_2$, is much harder to measure. Advanced waveform analysis shows that q for this system was in the range of 0.73 to 1.0. Calculate range of values for the masses m_1 (primary) and m_2 (secondary). (6pt)

$$M_{chirp} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

$$\therefore M_{chirp} = m_2 \sqrt[5]{\frac{q^3}{1+q}}$$

2.0

This is an increasing function of q

$$\therefore m_{2,min} = M_{chirp} 2^{1/5} = 1.37M_{\odot}$$

$$m_{2,max} = 1.60M_{\odot}$$

1.0

$$\text{Also, } M_{chirp} = \frac{m_1}{q} \sqrt[5]{\frac{q^3}{1+q}} = \frac{m_1}{\sqrt[5]{q^2(1+q)}}$$

2.0

$$\therefore m_{1,max} = M_{chirp} 2^{1/5} = 1.37M_{\odot}$$

$$m_{1,min} = 1.17M_{\odot}$$

1.0

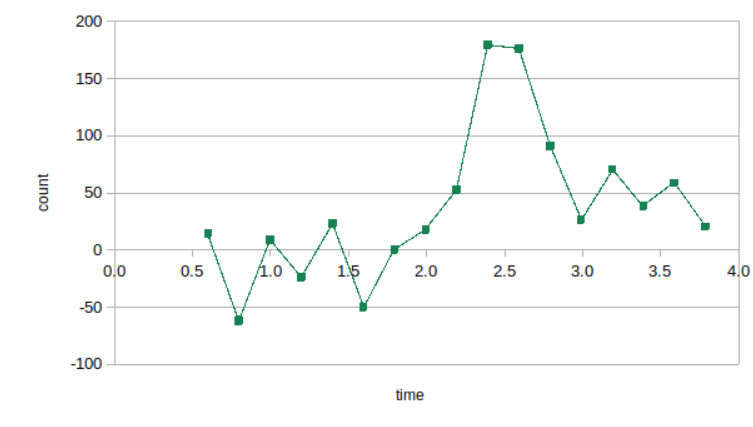
3 Speed of the gravitational wave

Gamma-ray burst GRB 170817A was observed by the Fermi Gamma-ray Burst Monitor to be almost simultaneous with gravitational wave event. The same neutron star merger was identified as the source of the signal. In the table below, the first column is time since the arrival of the gravitational wave peak, which happened on 2017 August 17 at 12:41:04 UTC. This peak symbolized the start of the merger of the neutron stars. The second column gives detector counts as measured in Fermi GBM. Background is already subtracted from the signal.

| $t(s)$ | event count (/s) |
|--------|------------------|
| 0.5981 | 15 |
| 0.7975 | -62 |
| 0.9968 | 9 |
| 1.1962 | -24 |
| 1.3956 | 24 |
| 1.5949 | -50 |
| 1.7943 | 0 |
| 1.9937 | 18 |
| 2.1930 | 53 |
| 2.3924 | 179 |
| 2.5918 | 176 |
| 2.7911 | 91 |
| 2.9905 | 26 |
| 3.1899 | 71 |
| 3.3892 | 38 |
| 3.5886 | 59 |
| 3.7880 | 21 |

(3.1) Plot event count over time on a millimeter paper.

(6pt)



(3.2) Estimate the delay Δt between start of the merger and start of the Gamma-ray Burst.

(1pt)

It appears that the event is starting from $\Delta t = 1.8\text{ s}$. Some students may also write $\Delta t = 2.2\text{ s}$ as that is when the signal is truly above fluctuations seen before the event. We will accept both the answers.

1.0

From this measurement it is possible to determine what is called the fractional speed difference during the trip.

$$\frac{\Delta v}{v_{EM}} = \frac{v_{GW} - v_{EM}}{v_{EM}}$$

(3.3) Express this quantity in terms of Δt and distance to the source D .

(1pt)

By simple substitution we can calculate that

$$\frac{\Delta v}{v_{EM}} = \frac{\Delta t v_{GW}}{D} \approx \frac{\Delta t c}{D}$$

1.0

(3.4) If we conservatively assume that the peak of the gravitational wave signal and the first GRB photons were emitted simultaneously, thus attributing the entire lag to faster

travel by the gravitational wave signal, this time difference provides an upper bound on Δv . Calculate this upper bound. **(1pt)**

$$\frac{\Delta v}{v_{EM}} = 4.2 \times 10^{-16} \text{ to } 5.1 \times 10^{-16}$$

1.0

(3.5) To obtain a lower bound on Δv , one can assume that the two signals were emitted at times differing by more than Δt with the faster EM signal making up some of the difference. Take maximum time delay as 10s and find lower bound. **(1pt)**

$$\Delta t' = \Delta t - 10 = -8.2 \text{ s to } 7.8 \text{ s}$$

From this time delay

$$\frac{\Delta v}{v_{EM}} = -1.9 \times 10^{-15} \text{ to } 1.8 \times 10^{-15}$$

1.0

Data Analysis - Galactic Surveys

In Cosmology, observations of galaxy groups is very important for the study and verification of complex theory models, Universe expansion, and its behaviour analysis. There are numerous telescopes used only for galactic surveys and making special statistical analysis. The product of all this are the catalogues of galaxies, which are the result of this observations and complex computational analysis.

Usually, galaxies form in groups inside very large dark matter Halos (that have a spherical shape). The most matter is concentrated in the center, thus having a central large-mass galaxy there is with high probability. In these problem, we will assume that all Halos have one central galaxy and several satellite galaxies.

As one can guess, all of these requires a large amount of computational power: To distinguish galaxies, process corrected coordinates with redshifts and other methods, etc. The complex statistical analysis involves calculating correlation functions for galaxy groups, which is making a histogram for distances between all possible two galaxy pairs. The main goal of this problem is to do theoretical statistical analysis of the observational data and figure out what kind of galaxy groups have been observed. Having a theoretical model that can deduce the characteristics of a galaxy groups with few observations is very useful for scientists and saves lots of computational time and possible errors following those kind of data analysis.

In Figure 1, you can see the probability densities of different galaxy groups. On X axis is the distance, and on Y axis is probability density. There are six galaxy cluster types (A-F) with corresponding densities. The goal of this problem will be to determine what kind of Dark matter Halo groups we are observing.

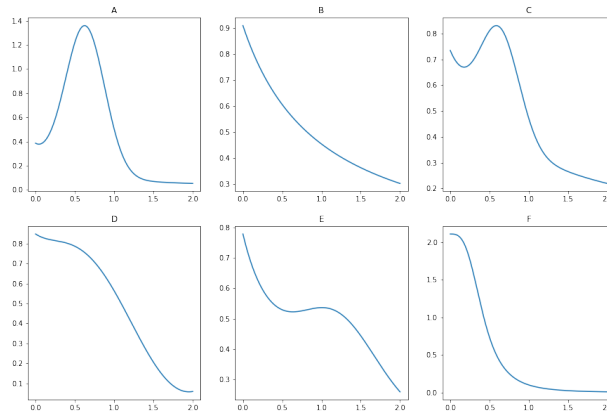


Figure 1: Probability density of all possible galaxy pairs (from different/same dark matter Halos)

Part 1 (35p)

You are given Table 1. that represents the data from observations. From other data and observations it was concluded that all these galaxies belong to the same dark matter halo. You can see the coordinates of satellite galaxies. The coordinate for the central galaxy is 8.300;6.200;1.100. Additional column represents the number of galaxies spotted very near to that points. For simplicity, you can assume that the coordinates are the same for them. These coordinates are given in Cartesian coordinate system center of which coincides to the observer on Earth.

| | x (MPC) | y (MPC) | z (MPC) | N (Number of galaxies) |
|----|---------|---------|---------|------------------------|
| 1 | 8.401 | 6.309 | 1.394 | 291 |
| 2 | 9.883 | 7.189 | 1.506 | 170 |
| 3 | 8.883 | 6.413 | 2.226 | 253 |
| 4 | 8.444 | 6.569 | 2.439 | 8 |
| 5 | 7.782 | 6.048 | -0.358 | 72 |
| 6 | 8.780 | 6.305 | 2.463 | 120 |
| 7 | 7.990 | 5.881 | 0.532 | 302 |
| 8 | 8.540 | 6.388 | 0.369 | 52 |
| 9 | 7.975 | 6.030 | 1.216 | 28 |
| 10 | 7.072 | 5.431 | 0.488 | 40 |
| 11 | 8.037 | 5.965 | 0.615 | 72 |
| 12 | 8.681 | 6.483 | 0.734 | 379 |
| 13 | 7.115 | 5.259 | 1.403 | 82 |
| 14 | 9.587 | 7.130 | 0.322 | 62 |
| 15 | 8.193 | 6.104 | 0.915 | 305 |
| 16 | 7.613 | 5.847 | 1.666 | 293 |

Before we start converting Cartesian coordinates to spherical we need to find appropriate local coordinate system with origin in the center of the halo.

- Plot all the pairs of Cartesian coordinates on the separate graphs. Given the fact that this halo is shaped like a disk find the direction of its normal. (4p)
- Write down the unit vectors of local coordinate system. Choose appropriate orientation and take X' axis such that coincides the line of sight. (3p)
- Convert the given Cartesian coordinates to the local $X'Y'Z'$ coordinates. Fill in the values of Table 1 (a separate sheet is given in the answer sheets): X' , Y' , Z' . (8p)
- Convert the local Cartesian coordinates into spherical (take the central galaxy as the center of the coordinate system, ϕ is calculated relative to X' axis and θ - relative to Z' axis). Fill in the other values of Table 1 (a separate sheet is given in the answer sheets): r , θ , ϕ . (8p)
- Plot the histogram (parameter dependence on total number of data points that fall in a given limits) with 8 bins of Satellite galaxy distances from the central Galaxy. Take 0.2-2 MPC as the histogram range. The histogram has to be also converted into probability density function. Note: The probability density function is the probability at a given point. You can avoid plotting two separate graphs by just noting other limits of Y axis in case of probability function. (6p)
- Theoretically the histogram is given by the following equation

$$H(r) = \frac{1}{A + B \cdot r} \quad (1)$$

Linearize the given dependence and find values A and B from the graph (6p)

Part 2 (20p)

It is well-known that while measuring distance to an object with redshift method (large relative velocity due to the expansion of the universe) errors might occur because of the motion of the galaxies. The aim of this chapter is to correct these errors.

Assume that the central galaxy has a mass of $10^{10} M_{Sun}$, all satellite galaxies move in circular orbits and interactions between them is negligible.

- (a) Use the data filled with you in Table 1. to plot the histogram of polar angle (take the width of histogram bins as 60 degrees). The angle distribution should be uniform, but it is not. Calculate the square deviation from the uniform value. (6p)
- (b) The galaxies originally in 60-120 and 240-300 bins would need most correction since they are the ones which move along the line of sight. Assume that the angular coordinates coincide with the center of the corresponding bin (90 and 270 in our case) and write the expression as a function of distance from the center which describes change of polar angles caused by circular motion of the galaxies. Hint: You can assume that the distance from the centre doesn't change significantly because of this correction. (6p)
- (c) Using the result of previous task correct the angles used in the last histogram. Plot the new histogram (probability dependence on distance) with the corrected data. Calculate the square deviation from the uniform value. (8p)

Part 3 (20p)

We have only calculated the central galaxy - satellite (from the same galaxy) histogram. It is time to move on to calculating other relations.

The same observations and data analysis has been carried out for different Halos and their central galaxies. The data for Figure 2. histogram only includes the distribution probability density function of the distances between central galaxies in different dark matter Halos.

Suppose we have 500 Halos, thus 500 central galaxies ($N_C = 500$). Take that the average number of satellite in one Halo is 1000 ($N_S = 1000$).

- (a) Calculate the following quantities: (3 p)
 1. N_{CC} , total number of central galaxy pairs
 2. N_{CS}^* , total number of central galaxy - satellite pairs in the same Halo
 3. N_{CS} , total number of central galaxy - satellite pairs in different Halos
 4. N_{SS}^* , total number of satellite - satellite galaxy pairs in the same Halo
 5. N_{SS} , total number of satellite - satellite galaxy pairs in different Halos

Below is given the table for probability density values for different distances between central galaxies in different dark matter Halos.

| r (MPC) | ρ |
|---------|--------|
| 0.3125 | 0.406 |
| 0.5375 | 0.808 |
| 0.7625 | 1.341 |
| 0.9875 | 1.134 |
| 1.2125 | 0.479 |
| 1.4375 | 0.143 |
| 1.6625 | 0.073 |
| 1.8875 | 0.060 |

We can calculate all probability densities using the given Center-Center data and Center-Satellite data which you calculated in part 1. The relations are following:

$$\rho_{SS}^*(r) = c_1 \rho_{CS}^*(r)^2 r \quad (2)$$

$$\rho_{CS}(r) = c_2 \rho_{CS}^*(r)^2 r \sqrt{\rho_{CC}(r) + 1} \quad (3)$$

$$\rho_{SS}(r) = c_3 \rho_{CS}^*(r)^2 r (\rho_{CC}(r) - 5)^2 \quad (4)$$

Where c_1 , c_2 and c_3 are normalization constant and should be found for each density.

- (b) Calculate and plot $\rho_{SS}^*(r)$, $\rho_{CS}(r)$ and $\rho_{SS}(r)$. (10p)
- (c) Plot the final probability density of all galaxy pairs. Which one of 6 graphs on Figure 1 is most similar to you plot? (7p)

Solutions - Part 1

Part (a)

If we plot given coordinates in XY, XZ and YZ planes we'll see the following pictures. See Figures 2, 3 and 4 (0.5p for each graph). By observing Figure 2 we can deduce that the group is edge-on and its normal lies in XY plane. X' axis is along line of sight so it can be calculated by normalizing $8.3\hat{i} + 6.2\hat{j} + 1.1\hat{k}$. So

$$\hat{x}' = 0.79667866\hat{i} + 0.59510936\hat{j} + 0.10558392\hat{k} \quad (5)$$

(0.5p) Z' axis lays in XY plane and is perpendicular to X' so it can be calculated by normalizing $\hat{x}' \times \hat{k}$

$$\hat{z}' = 0.59845448\hat{i} - 0.80115681\hat{j} \quad (6)$$

(2p)

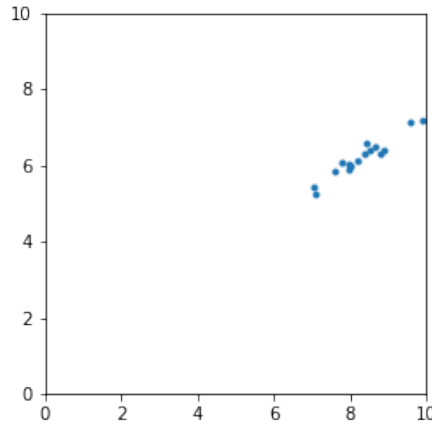


Figure 2: XY viewpoint of a cluster from observational site

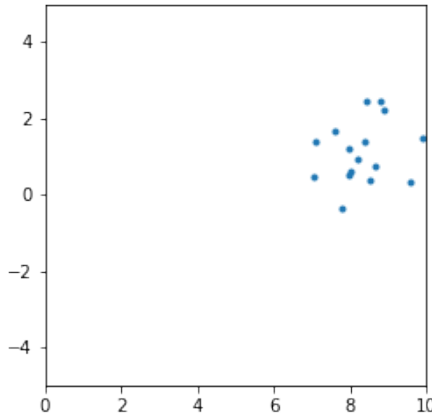


Figure 3: XZ viewpoint of a cluster from observational site

Part (b)

One can choose x' as line of sight (found in part(a)) (1p):

$$\hat{x}' = 0.79667866\hat{i} + 0.59510936\hat{j} + 0.10558392\hat{k} \quad (7)$$

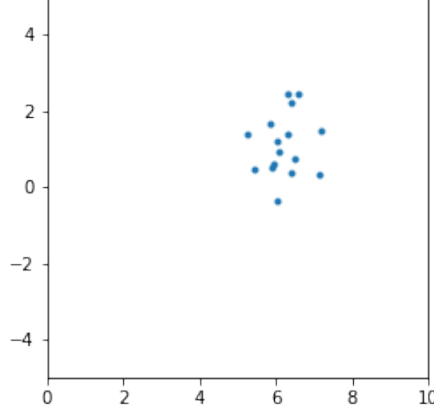


Figure 4: YZ viewpoint of a cluster from observational site

z' can be chosen as halo disk normal (found in part(a))(1p):

$$\hat{z}' = 0.59845448\hat{i} - 0.80115681\hat{j} \quad (8)$$

And finally for y' (1p):

$$\hat{y}' = \hat{x}' \times \hat{z}' = 0.08458928\hat{i} + 0.06318717\hat{j} - 0.9944104\hat{k} \quad (9)$$

Part (c)

Each coordinate in $X'Y'Z'$ system can be found by following equations

$$x' = \hat{x}' \cdot (\vec{r} - \vec{r}_c) \quad (10)$$

$$y' = \hat{y}' \cdot (\vec{r} - \vec{r}_c) \quad (11)$$

$$z' = \hat{z}' \cdot (\vec{r} - \vec{r}_c) \quad (12)$$

Where \hat{x}' , \hat{y}' and \hat{z}' are unit vectors of local $X'Y'Z'$ system, \vec{r} is coordinate of a galaxy and \vec{r}_c is coordinate of the center.

This values are calculated in the following table (0.5p for each correct x , y' , z' tuple):

| | $x - x_c$ | $y - y_c$ | $z - z_c$ | x' | y' | z' |
|----|-----------|-----------|-----------|-----------|-----------|-----------|
| 1 | 0.101124 | 0.109271 | 0.294476 | 0.176684 | -0.277372 | -0.027025 |
| 2 | 1.582909 | 0.989392 | 0.405895 | 1.892722 | -0.207212 | 0.154641 |
| 3 | 0.582735 | 0.213285 | 1.126426 | 0.710112 | -1.057359 | 0.177866 |
| 4 | 0.144328 | 0.369381 | 1.338567 | 0.476137 | -1.295537 | -0.209558 |
| 5 | -0.517770 | -0.151687 | -1.458202 | -0.656729 | 1.396668 | -0.188336 |
| 6 | 0.480059 | 0.105110 | 1.362674 | 0.588881 | -1.307808 | 0.203084 |
| 7 | -0.309855 | -0.319116 | -0.568033 | -0.496739 | 0.518483 | 0.070227 |
| 8 | 0.239541 | 0.188218 | -0.730736 | 0.225694 | 0.758807 | -0.007438 |
| 9 | -0.324800 | -0.169551 | 0.116378 | -0.347375 | -0.153915 | -0.058541 |
| 10 | -1.227659 | -0.768684 | -0.612379 | -1.500158 | 0.456538 | -0.118862 |
| 11 | -0.263209 | -0.234869 | -0.485227 | -0.400698 | 0.445410 | 0.030649 |
| 12 | 0.381200 | 0.282515 | -0.365581 | 0.433222 | 0.413634 | 0.001792 |
| 13 | -1.185021 | -0.941303 | 0.302593 | -1.472310 | -0.460620 | 0.044950 |
| 14 | 1.286640 | 0.930147 | -0.777889 | 1.496445 | 0.941150 | 0.024802 |
| 15 | -0.106760 | -0.095804 | -0.184997 | -0.161600 | 0.168879 | 0.012863 |
| 16 | -0.686865 | -0.352750 | 0.566244 | -0.697350 | -0.643470 | -0.128449 |

Part (d)

Spherical coordinates can be calculated by following relations:

$$r = \sqrt{x'^2 + y'^2 + z'^2} \quad (13)$$

$$\phi = \arctan \frac{y'}{x'} \quad (14)$$

and signs of x' and y'

$$\theta = \arccos \frac{z'}{r} \quad (15)$$

Calculated values are given in the following table (0.5p for each correct r, ϕ, θ tuple):

| | x' | y' | z' | r | ϕ | θ |
|----|-----------|-----------|-----------|----------|----------|----------|
| 1 | 0.176684 | -0.277372 | -0.027025 | 0.329974 | 5.279566 | 1.652789 |
| 2 | 1.892722 | -0.207212 | 0.154641 | 1.910301 | 6.174141 | 1.489757 |
| 3 | 0.710112 | -1.057359 | 0.177866 | 1.286042 | 5.303793 | 1.432047 |
| 4 | 0.476137 | -1.295537 | -0.209558 | 1.396079 | 5.064586 | 1.721471 |
| 5 | -0.656729 | 1.396668 | -0.188336 | 1.554814 | 2.010330 | 1.692226 |
| 6 | 0.588881 | -1.307808 | 0.203084 | 1.448581 | 5.135477 | 1.430138 |
| 7 | -0.496739 | 0.518483 | 0.070227 | 0.721461 | 2.334780 | 1.473302 |
| 8 | 0.225694 | 0.758807 | -0.007438 | 0.791695 | 1.281697 | 1.580191 |
| 9 | -0.347375 | -0.153915 | -0.058541 | 0.384430 | 3.558678 | 1.723672 |
| 10 | -1.500158 | 0.456538 | -0.118862 | 1.572587 | 2.846171 | 1.646452 |
| 11 | -0.400698 | 0.445410 | 0.030649 | 0.599906 | 2.303400 | 1.519685 |
| 12 | 0.433222 | 0.413634 | 0.001792 | 0.598981 | 0.762273 | 1.567805 |
| 13 | -1.472310 | -0.460620 | 0.044950 | 1.543336 | 3.444801 | 1.541667 |
| 14 | 1.496445 | 0.941150 | 0.024802 | 1.767973 | 0.561416 | 1.556768 |
| 15 | -0.161600 | 0.168879 | 0.012863 | 0.234094 | 2.334171 | 1.515821 |
| 16 | -0.697350 | -0.643470 | -0.128449 | 0.957522 | 3.886829 | 1.705349 |

Part (e)

Using radii calculated above and number of galaxies for each data (rewritten below) one can calculate number of galaxies in each bin. Range of bin 1 is (0.2, 0.425), bin 2 - (0.425, 0.65) and so on bin 8 - (1.775, 2.0).

| | r | n |
|----|----------|-----|
| 1 | 0.329974 | 291 |
| 2 | 1.910301 | 170 |
| 3 | 1.286042 | 253 |
| 4 | 1.396079 | 8 |
| 5 | 1.554814 | 72 |
| 6 | 1.448581 | 120 |
| 7 | 0.721461 | 302 |
| 8 | 0.791695 | 52 |
| 9 | 0.384430 | 28 |
| 10 | 1.572587 | 40 |
| 11 | 0.599906 | 72 |
| 12 | 0.598981 | 379 |
| 13 | 1.543336 | 82 |
| 14 | 1.767973 | 62 |
| 15 | 0.234094 | 305 |
| 16 | 0.957522 | 293 |

Correct values for each bin are: (0.5p for each correct n)

| | Center of the bin | Number of the galaxies |
|---|-------------------|------------------------|
| 1 | 0.312500 | 624 |
| 2 | 0.537500 | 451 |
| 3 | 0.762500 | 354 |
| 4 | 0.987500 | 293 |
| 5 | 1.212500 | 253 |
| 6 | 1.437500 | 210 |
| 7 | 1.662500 | 174 |
| 8 | 1.887500 | 170 |

Corresponding histogram is given on Figure 5. (1p)

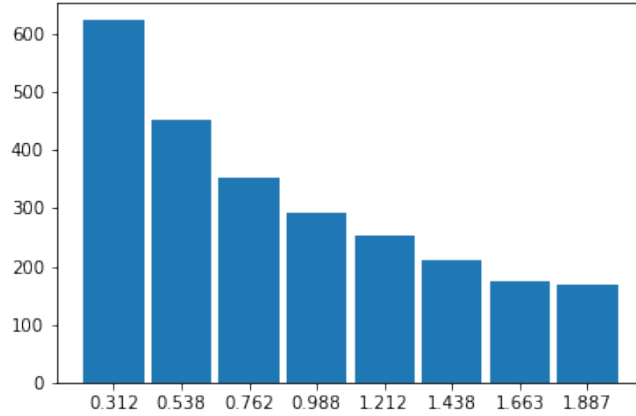


Figure 5: Histogram: Number of galaxies dependence on distance (MPC), with a bin size equal to 0.225 MPC

Histogram data as a table:

In order to get probability density function each value of histogram must be divided by its integral on the whole region. The integral can be approximated by finding the total area of the rectangles:

$$I = \sum_{n=1}^8 H(r_i)d = 569.025 \quad (16)$$

Where $d = 0.225$ is the size of a bin. After dividing each data by I we get (0.25p for each correct probability density):

| | Center of the bin | Number of the galaxies | Probability density |
|---|-------------------|------------------------|---------------------|
| 1 | 0.312500 | 624 | 1.096613 |
| 2 | 0.537500 | 451 | 0.792584 |
| 3 | 0.762500 | 354 | 0.622117 |
| 4 | 0.987500 | 293 | 0.514916 |
| 5 | 1.212500 | 253 | 0.444620 |
| 6 | 1.437500 | 210 | 0.369052 |
| 7 | 1.662500 | 174 | 0.305786 |
| 8 | 1.887500 | 170 | 0.298757 |

Plotting this data gives the graph on Figure 6 (1p)

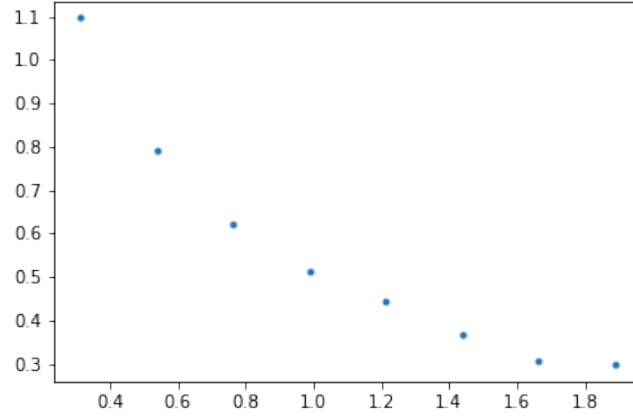


Figure 6: Probability density of center-satellite distances in the same halo

Part (f)

Since histogram equation has following form:

$$H(r) = \frac{1}{A + B \cdot r} \quad (17)$$

We should calculate $1/H(r)$ which gives the following values (0.125 for each correct $1/H$) :

| | r | H(r) | 1/H(r) |
|---|----------|------|----------|
| 1 | 0.312500 | 624 | 0.001603 |
| 2 | 0.537500 | 451 | 0.002217 |
| 3 | 0.762500 | 354 | 0.002825 |
| 4 | 0.987500 | 293 | 0.003413 |
| 5 | 1.212500 | 253 | 0.003953 |
| 6 | 1.437500 | 210 | 0.004762 |
| 7 | 1.662500 | 174 | 0.005747 |
| 8 | 1.887500 | 170 | 0.005882 |

Plotting this data gives Figure 7

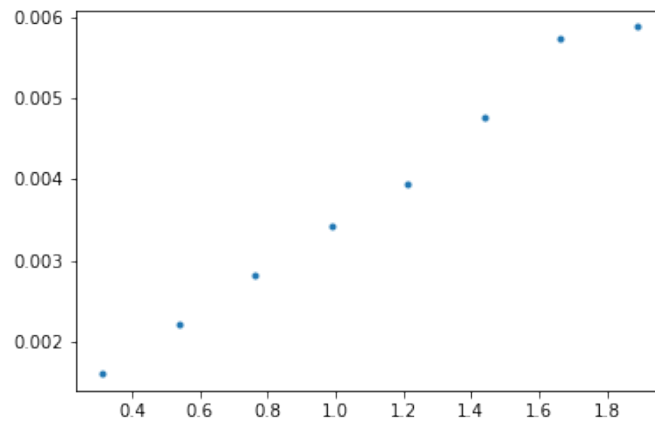


Figure 7: Linearization for regression

Finding the slope and the intercept of this line (by graph or any other method) gives

Following values:

$$A = 0.00065977; B = 0.00285494 \quad (18)$$

2.5p for B in range 0.002-0.003 2.5p for A in range 0.0006-0.0008

Solutions - Part 2

Part (a)

We can ϕ values from Part 1.c to distribute them in 60 degree bins. In bin 1 we have number of galaxies which fall in (0, 60) degree range, in bin 2 - (60, 120) and so on bin 6 - (300, 360).

Part 1.c data in degrees:

| | Angle | Number of galaxies |
|----|------------|--------------------|
| 1 | 302.496846 | 291 |
| 2 | 353.752245 | 170 |
| 3 | 303.884926 | 253 |
| 4 | 290.179429 | 8 |
| 5 | 115.183442 | 72 |
| 6 | 294.241133 | 120 |
| 7 | 133.773013 | 302 |
| 8 | 73.435830 | 52 |
| 9 | 203.897219 | 28 |
| 10 | 163.073590 | 40 |
| 11 | 131.975076 | 72 |
| 12 | 43.675004 | 379 |
| 13 | 197.372555 | 82 |
| 14 | 32.166754 | 62 |
| 15 | 133.738158 | 305 |
| 16 | 222.698869 | 293 |

We get the following data for the histogram (0.5p for each correct n):

| | Center of the bin | Number of galaxies |
|---|-------------------|--------------------|
| 1 | 30 | 441 |
| 2 | 90 | 124 |
| 3 | 150 | 719 |
| 4 | 210 | 403 |
| 5 | 270 | 128 |
| 6 | 330 | 714 |

Uniform value is total number of galaxies divides by 6 which is 421.5. Square deviation is 241° (1p)

Graph is given on Figure 8 (2p)

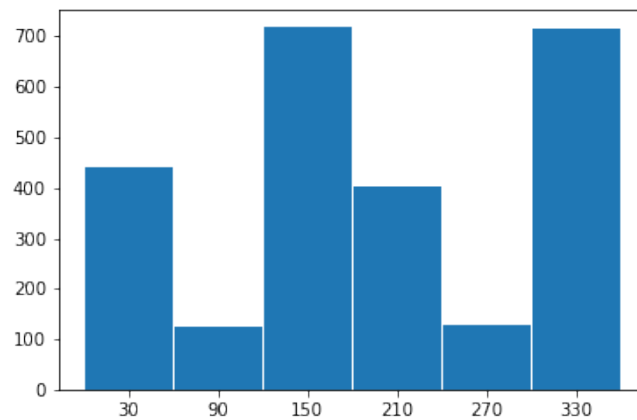


Figure 8: Histogram: Number of galaxies dependence on ϕ angle, with a bin size equal to 60°

Part (b)

A galaxy with the orbital radius r around central mass M has velocity (2p):

$$v = \sqrt{\frac{GM}{r}} \quad (19)$$

This will cause to give the following error in distance along line of sight (2p):

$$d = \frac{v}{H} = \frac{1}{H} \sqrt{\frac{GM}{r}} \quad (20)$$

Angle difference for galaxies at 90 and 270 degrees will be (2p):

$$\alpha = \arctan \frac{d}{r} = \arctan \frac{1}{H} \sqrt{\frac{GM}{r^3}} \quad (21)$$

For detailed drawing see Figure 9

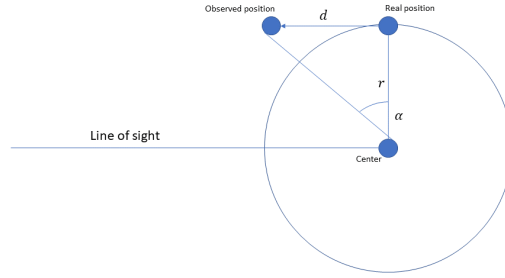


Figure 9: Drawing

Part (c)

Galaxies originally in bins of range (60, 120) and (240, 300) might move to (120, 180) and (300, 360) respectively. So we should subtract correction value found in previous task to the galaxies in bins 2 and 5 to see if they move to bins 1 and 4 respectively. (0.25p for each correct corrected angle)

| | Original angle | Corrected angle | Number of galaxies |
|----|----------------|-----------------|--------------------|
| 1 | 302.494665 | 276.136451 | 291 |
| 2 | 353.752069 | 351.714594 | 170 |
| 3 | 303.884728 | 300.199512 | 253 |
| 4 | 290.178642 | 290.178642 | 8 |
| 5 | 115.184577 | 115.184577 | 72 |
| 6 | 294.241835 | 294.241835 | 120 |
| 7 | 133.776280 | 125.061576 | 302 |
| 8 | 73.437747 | 73.437747 | 52 |
| 9 | 203.899065 | 203.899065 | 28 |
| 10 | 163.073490 | 160.346665 | 40 |
| 11 | 131.977481 | 120.548626 | 72 |
| 12 | 43.676971 | 43.676971 | 379 |
| 13 | 197.372956 | 197.372956 | 82 |
| 14 | 32.167948 | 32.167948 | 62 |
| 15 | 133.743596 | 94.077612 | 305 |
| 16 | 222.694636 | 222.694636 | 293 |

New corrected distribution looks like this (0.3p for each correct n):

| | Center of the bin | Number of galaxies |
|---|-------------------|--------------------|
| 1 | 30 | 441 |
| 2 | 90 | 429 |
| 3 | 150 | 414 |
| 4 | 210 | 403 |
| 5 | 270 | 419 |
| 6 | 330 | 423 |

In this case square deviation equals to 12°

(1p) And relevant graph is given on Figure 10 (1.2p)

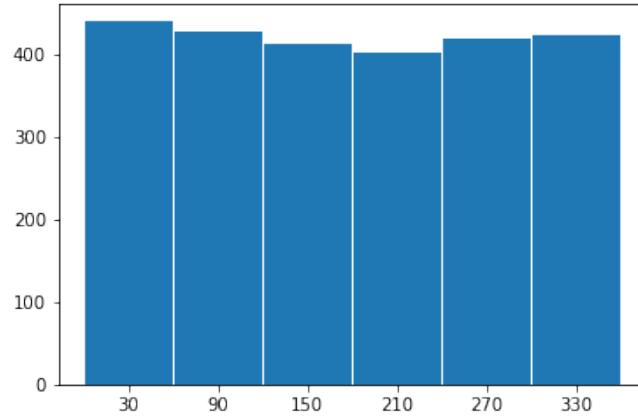


Figure 10: Histogram: Corrected Number of galaxies dependence on ϕ angle, with a bin size equal to 60°

Solutions - Part 3

Part (a)

With simple combinatorics one can get (0.6p for each correct N):

$$N_{CC} = N_C(N_C - 1) = 249500 \quad (22)$$

$$N_{CS}^* = N_C N_S = 500000 \quad (23)$$

$$N_{CS} = N_C(N_C - 1)N_S = 249500000 \quad (24)$$

$$N_{SS}^* = N_C N_S(N_S - 1) = 499500000 \quad (25)$$

$$N_{SS} = N_C N_S(N_C - 1)N_S = 249500000000 \quad (26)$$

Part (b)

We can calculate all probability densities using center-center and center-satellite densities. Non-normalized values (with $c_1 = c_2 = c_3 = 1$) are calculated in the following table (0.5p for each correct tuple):

| | r | ρ_{CS}^* | ρ_{CC} | ρ_{SS}^* | ρ_{SS} | ρ_{CS} |
|---|----------|---------------|-------------|---------------|-------------|-------------|
| 1 | 0.312500 | 1.096613 | 0.406141 | 0.375800 | 17.365137 | 0.975747 |
| 2 | 0.537500 | 0.792584 | 0.807863 | 0.337652 | 12.992898 | 0.994072 |
| 3 | 0.762500 | 0.622117 | 1.340510 | 0.295110 | 8.653487 | 0.988565 |
| 4 | 0.987500 | 0.514916 | 1.134011 | 0.261824 | 8.568351 | 0.837480 |
| 5 | 1.212500 | 0.444620 | 0.478630 | 0.239696 | 10.729180 | 0.638199 |
| 6 | 1.437500 | 0.369052 | 0.143154 | 0.195787 | 10.112507 | 0.458355 |
| 7 | 1.662500 | 0.305786 | 0.073165 | 0.155452 | 8.262279 | 0.352612 |
| 8 | 1.887500 | 0.298757 | 0.060971 | 0.168470 | 8.998531 | 0.379962 |

For normalization (same way as in Part 1.d) we get $c_1 = 2.18592$; $c_2 = 0.79304$; $c_3 = 0.0516$. Multiplying relevant densities by this values gives (0.5p for each correct normalized tuple):

| | r | ρ_{SS}^* | ρ_{SS} | ρ_{CS} |
|---|----------|---------------|-------------|-------------|
| 1 | 0.312500 | 0.822854 | 0.900750 | 0.770962 |
| 2 | 0.537500 | 0.739325 | 0.673957 | 0.785441 |
| 3 | 0.762500 | 0.646175 | 0.448866 | 0.781089 |
| 4 | 0.987500 | 0.573292 | 0.444450 | 0.661713 |
| 5 | 1.212500 | 0.524839 | 0.556535 | 0.504257 |
| 6 | 1.437500 | 0.428697 | 0.524548 | 0.362158 |
| 7 | 1.662500 | 0.340380 | 0.428574 | 0.278607 |
| 8 | 1.887500 | 0.368883 | 0.466764 | 0.300217 |

Plots are given on Figures 11 - 14 (0.5p for each plot).

Part (c)

Total probability density is weighted sum of all densities (4p):

$$\rho \sim \frac{N_{CC}\rho_{CC} + N_{CS}^*\rho_{CS}^* + N_{CS}\rho_{CS} + N_{SS}^*\rho_{SS}^* + N_{SS}\rho_{SS}}{N_{CC} + N_{CS}^* + N_{CS} + N_{SS}^* + N_{SS}} \quad (27)$$

Calculated values and normalized densities are given in the following table:

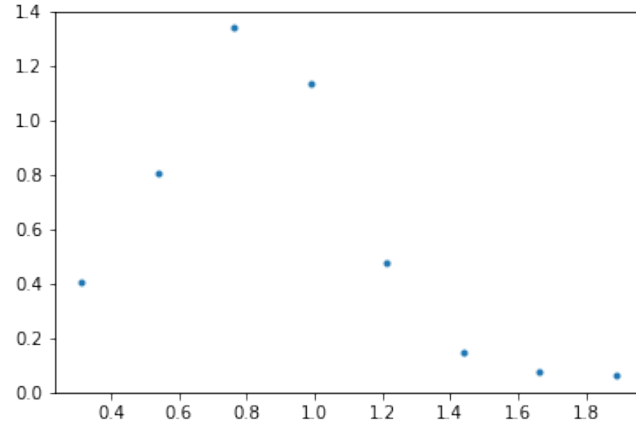


Figure 11: Probability density of center-center distances in for different halos

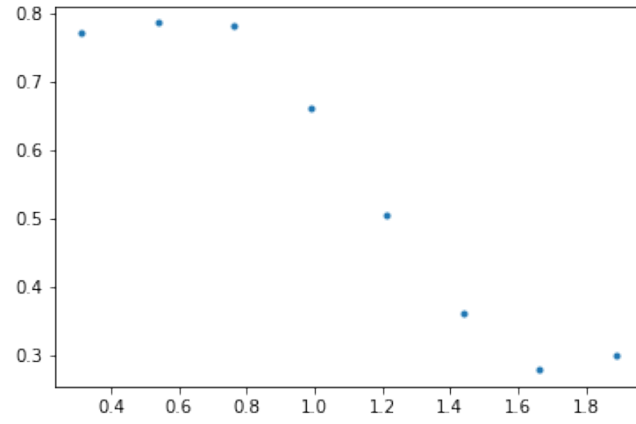


Figure 12: Probability density of center-satellite distances in the different halos

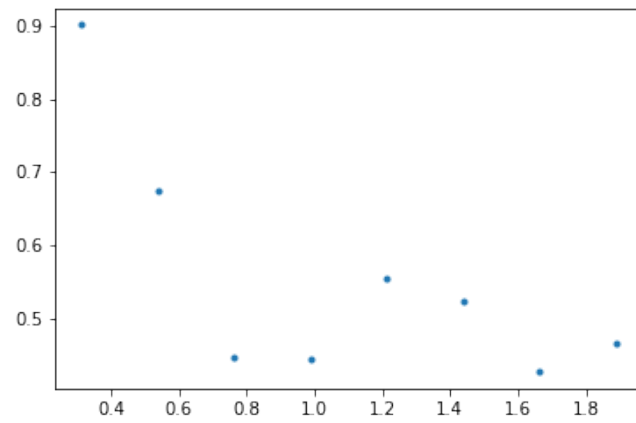


Figure 13: Probability density of satellite-satellite distances in the different halos

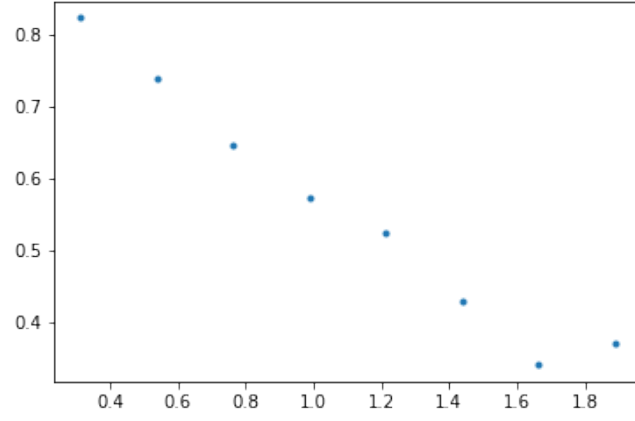


Figure 14: Probability density of satellite-satellite distances in the same halo

| | r | Calculated ρ | Normalized ρ |
|---|----------|-------------------|-------------------|
| 1 | 0.312500 | 0.004212 | 0.843662 |
| 2 | 0.537500 | 0.003606 | 0.722433 |
| 3 | 0.762500 | 0.002967 | 0.594357 |
| 4 | 0.987500 | 0.002693 | 0.539524 |
| 5 | 1.212500 | 0.002663 | 0.533361 |
| 6 | 1.437500 | 0.002265 | 0.453652 |
| 7 | 1.662500 | 0.001813 | 0.363239 |
| 8 | 1.887500 | 0.001968 | 0.394216 |

And final plot looks like this (Figure 15) (2p)

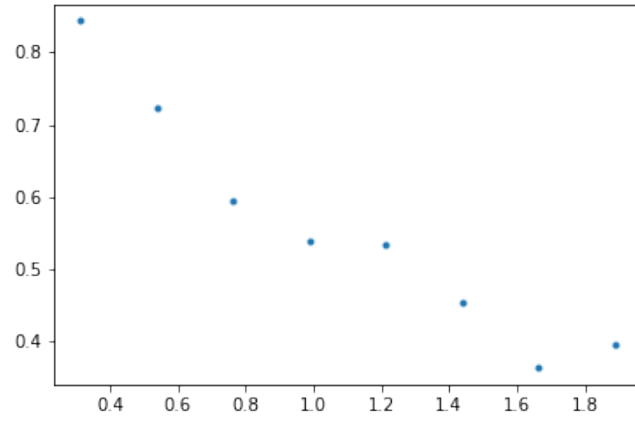


Figure 15: Final probability density

Most similar to this density is plot E on Figure 1 (1p).